The veering census

Saul Schleimer
University of Warwick
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joint work with
Henry Segerman
Organizing Committee

- Jayadev Athreya  
  University of Washington
- Alexander Holroyd  
  Churchill College and Statistical Laboratory at University of Cambridge
- Sarah Koch  
  University of Michigan, Ann Arbor

Abstract

This workshop will focus on the theoretical insights developed via illustration, visualization, and computational experiment in dynamical systems and probability theory. Some topics from complex dynamics include: dynamical moduli spaces and their dynamically-defined subvarieties, degenerations of dynamical systems as one moves toward the boundary of moduli space, and the structure of algebraic data coming from a family of dynamical systems. In classical dynamical systems, some topics include: flows on hyperbolic spaces and Lorentz attractors, simple physical systems like billiards in two and three dimensional domains, and flows on moduli spaces. In probability theory, the workshop features: random walks and continuous time random processes like Brownian motion, SLE, and scaling limits of discrete systems.
Tools and applications
Example
The (-2, 3, 7) pretzel knot
The (-2, 3, 7) pretzel knot
Triangulations
Veering tetrahedra

- Red fan
- Blue fan
- Red on top toggle
- Blue on top toggle
The \((-2, 3, 7)\) pretzel knot
Veering triangulations are rare
The **SnapPea** census (up to seven tetrahedra)

- 4,815 orientable triangulations
- All are geometric so all have strict angle structures
- 13,599 taut angle structures on these triangulations
- 158 veering structures (on 151 triangulations)
Another way to sample triangulations: explore the *Pachner graph* of triangulations of a manifold.

(Matveev (1987), Piergallini (1988)) The Pachner graph is connected under 2-3 and 3-2 moves.

In the “ceiling 9” subgraph of the Pachner graph for the \((-2,3,7)\) pretzel knot complement:

<table>
<thead>
<tr>
<th>Triangulations</th>
<th>1,222,561</th>
<th>100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Admit a taut angle structure</td>
<td>153,474</td>
<td>12.6%</td>
</tr>
<tr>
<td>Admit a strict angle structure</td>
<td>2,365</td>
<td>0.193%</td>
</tr>
<tr>
<td>Admit a veering structure</td>
<td>1</td>
<td>0.0000818%</td>
</tr>
</tbody>
</table>
Censuses
Censuses in low-dimensional topology

- **Knots**: Tait, Little, Conway, Rolfsen, Hoste—Thistlewaite—Weeks, Champanerkar—Kofman—Mullen, …

- **Manifolds**: Weeks, Matveev, Callahan—Hildebrand—Weeks, Thistlewaite, Burton, …

- **Triangulations of $S^3$**: Burton

- **Monodromies**: Bell-Hall-S, Bell
The veering census
Ideal solid tori
red fan
blue fan
red on top toggle
blue on top toggle
Solid tori glue to each other along rhombuses on their boundaries, matching edge colours.

To build our census of transverse veering structures, we try all such gluings.

We get a transverse veering structure if the total angle at each edge is $2\pi$. 
The (-2, 3, 7) pretzel knot
The number of veering structures approximately doubles every time we increase the number of tetrahedra by one.
## The veering census

<table>
<thead>
<tr>
<th>tetrahedra</th>
<th>veering</th>
<th>non-geometric</th>
<th>non-layered</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>0</td>
<td>4</td>
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<tr>
<td>6</td>
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<tr>
<td>8</td>
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<td>1</td>
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<tr>
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<td>110</td>
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<td>21283</td>
<td>234</td>
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</tr>
<tr>
<td>16</td>
<td>43763</td>
<td>503</td>
<td>18157</td>
</tr>
</tbody>
</table>

Census available at https://math.okstate.edu/people/segerman/veering.html
The veering census

Conjectures:

• The number of veering triangulations grows super-exponentially with $n$.

• The percentage of veering triangulations that are geometric tends to zero as $n$ tends to infinity.

• The percentage of veering triangulations that are layered tends to zero as $n$ tends to infinity.

• Any hyperbolic cusped three-manifold admits only finitely many veering triangulations (and some have none).
A leaf carried by the stable branched surface for the veering triangulation of the figure 8 knot complement. The leaf is decomposed into sectors, and then into normal disks.